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Coulomb gas representation of quantum Hall effect on Riemann surfaces

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Abstract. Using the correlation function of the chiral vertex operators of the Coulomb gas model, we find the Laughlin wavefunctions of the quantum Hall effect, with filling factor $\nu = 1/m$, on Riemann sufaces with a Poincaré metric. The same is done for quasihole wavefunctions. We also discuss their plasma analogy.

1. Introduction

The study of the behaviour of electrons living on a two-dimensional surface and interacting with a constant magnetic field orthogonal to the surface is one of the most important parts of physics, and is known as the quantum Hall effect (QHE). When the surface is plane, both integer QHE and fractional QHE have been observed experimentally, and these have been described by Landau- and Jastrow-type wavefunctions, respectively [1]. Also by introducing new types of many body condensates which carry fractional charge, i.e. anyons, a unified picture of integer and fractional QHE has been given. Furthermore, as it has been stressed by Laughlin, the Laughlin–Jastrow wavefunctions, both for electrons and anyons have a natural interpretation in terms of a two-dimensional plasma of charges, interacting by Coulomb forces and embedded in a uniform neutralizing background [1].

A particularly intriguing and interesting case occurs when the two-dimensional surface is a Riemann surface of higher genus. Although it is not accessible experimentally, the problem of the physics on Riemann surfaces has a deep relation with some interesting problems, like the occurrence of chaos in the surface with negative curvature [2], and recent developments in the theory of surfaces, for example, surface moduli and the vector bundle defined on the moduli [3]. Until now, the QHE on different non-flat surfaces have been studied, for example, on the sphere [4], torus [5], and on the hyperbolic plane [6, 7]. A recent and detailed investigation was performed in [8], in which the Landau and Laughlin levels were studied on Riemann surface with some particular metrics.

In [8] the authors showed that the wavefunctions consist of two parts. A holomorphic part which is independent of the metric and a known metric-dependent function. For Landau levels, they showed that the holomorphic part is the Slater determinant of sections of the holomorphic line bundle, and for the Laughlin states they obtained an ansatz for the holomorphic part and showed that this function is the determinant of the holomorphic sections of a vector bundle. This vector bundle is the tensor product of a line bundle and a flat vector bundle of rank m ($\nu = 1/m$ is the filling factor).

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Now, there is another approach for studying the QHE in which the conformal symmetry of the QHE is used to calculate the different quantities. There are several pieces of evidence for the existence of this symmetry. For example, in [9] it was shown that the Laughlin wavefunctions are related to the conformal blocks of two-dimensional conformal field theories (CFTs). In the same paper, the fractional statistics of quasiparticles was related to the braiding properties of the vertex operators of the Coulomb gas model, which has conformal symmetry. In [10] it was shown that the Halperin–Haldane singlet quantum Hall effect wavefunction can be split into two parts. One part is related to a state describing a one-component plasma (OCP) system, and the other part behaves like a conformal block of primary fields of the $su(2)$ Wess–Zumino–Witten model. Also in [11] it was shown that the holomorphic part of the Laughlin wavefunction on the torus can be obtained by the correlation function of the Coulomb gas vertices. Moreover, it was pointed out that the Coulomb gas approach and the OCP description of the Laughlin wavefunction are consistent.

In this paper we will study the QHE on an arbitrary Riemann surface, in the context of CFT. Our purposes for this investigation are as follows. First, to see whether this relation between the QHE and the Coulomb gas model can be generalized to general Riemann surfaces. Second, as we will see, this approach is much easier than those considered in [8]. Third, our approach can easily be generalized to the case where anyons also exist.

The plan of this paper is as follows. In section 2 we will discuss in brief the relation between the OCP and the CFT description of the QHE. In section 3, by using the correlation function of the Coulomb gas vertices on a Riemann surface (derived in [12–15]), we obtain the holomorphic part of the Laughlin wavefunction. To do this, we must determine the parameters of the corresponding Coulomb gas model appropriately. We will also discuss the different aspects of this equivalence. In section 4, we obtain an expression for quasiholes wavefunctions and determine their charges in this context.

2. The QHE on the plane

To obtain insight into the relation between the QHE and CFT, let us recall the Laughlin wavefunction. It was shown by Laughlin [1] that the wavefunction of the QHE is the many particle wavefunction which looks like

$$\psi(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = \prod_{i=1}^{N} \exp\left(-\int A_{\bar{z}_i}\, \mathrm{d}\bar{z}_i\right) F(z_1, \ldots, z_N) \tag{1}$$

where $z_j = x_j + \mathrm{i}y_j$ is the position of the jth particle and $A_{\bar{z}} = \frac{1}{2}(A_x + \mathrm{i}A_y)$ is the gauge potential. The main purpose of theoretical investigations of the QHE is to determine the holomorphic function $F(z_1, \ldots, z_N)$ which must obey the Fermi statistics. Laughlin chose this function to be the eigenfunction of the angular momentum, and showed that $\prod_{i<j}^{N}(z_i - z_j)^m$ is an appropriate function for a filling factor $\nu = 1/m$. The final result, in $A_x = -\frac{1}{2}By$ and $A_y = \frac{1}{2}Bx$ gauge, is

$$\psi(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = \prod_{i<j}^{N}(z_i - z_j)^m \prod_{i=1}^{N} \mathrm{e}^{-\frac{1}{4}|z_i|^2}. \tag{2}$$

By introducing the classical potential energy U through $|\psi|^2 = \mathrm{e}^{-\beta U}$, where β^{-1} is an arbitrary effective temperature, Laughlin showed that this system is equivalent to a two-dimensional plasma of particles with electric charge m, interacting by Coulomb forces and embedded in a uniform neutralizing background.

Now the interesting point is that this OCP description of the QHE can also be achieved by considering the Coulomb gas model. This model is a free massless scalar field modified with a background charge at infinity. The two-point function of these fields [16] satisfies

$$\partial_z \partial_{\bar{z}} \langle \Phi(z, \bar{z}) \Phi(w, \bar{w}) \rangle = \pi \delta^2(z, w) \tag{3}$$

and $\Phi(z, \bar{z})$ splits into holomorphic and antiholomorphic parts. Note that this equation is simply the Laplace equation for the Coulomb interaction. If $\varphi(z)$ is the holomorphic part of $\Phi(z, \bar{z})$, then the expectation value of the product of the vertex operators $: \mathrm{e}^{\mathrm{i}q\varphi(z_i)} :$ is

$$F(z_1, \ldots, z_N) = \langle : \mathrm{e}^{\mathrm{i}q_1\varphi(z_1)} : \cdots : \mathrm{e}^{\mathrm{i}q_N\varphi(z_N)} : \rangle = \exp\left(-q_i q_j \sum_{i<j}^{N} \langle \varphi(z_i) \varphi(z_j) \rangle \right). \tag{4}$$

Now as on the plane $\langle \varphi(z_i)\varphi(z_j) \rangle = -\ln(z_i - z_j)$, which is the Coulomb potential in two dimensions, the holomorphic part of (2) is recovered by choosing $q_i = \sqrt{m}$ for all i (as the particles are identical) and $\beta = 1/m$. In this manner the Coulomb gas and the QHE relate to each other on the plane, i.e. each vertex corresponds to an electron and its conformal charge $\sqrt{m}$ relates to the electric charge of plasma particles. In summary, as the Coulomb gas model is effectively a theory of the Coulomb interaction and the QHE has a plasma analogy which again is based on the Coulomb interaction, therefore the results of these two theories *coincide*. In section 3 we will use this correspondence, and also the braiding properties of vertex operators, to find the Laughlin wavefunctions on Riemann surfaces.

3. The Coulomb gas approach to the QHE on a Riemann surface with a Poincaré metric

Now we study the QHE on a two-dimensional compact and orientable Riemann surface Σ. On this surface, the charged particles interact with the constant orthogonal magnetic field produced by the monopoles. We choose, as in [8], the Poincaré metric $g_{z\bar{z}} = y^{-2}$. The simply connected covering space of Σ is the upper half-plane H and $\Sigma = H/\Gamma$, where Γ is a discrete subgroup of the isometry group of H. Γ is generated by Fuchsian transformations around a canonical homology basis. For a covariantly constant magnetic field B, and in the symmetric gauge $A_z = A_{\bar{z}} = \frac{1}{2}By$, the one-particle Hamiltonian [8] is

$$H = -g^{z\bar{z}} D\bar{D} + B/4 \tag{5}$$

where $D = \partial - \frac{1}{2}B\partial \ln g_{z\bar{z}}$ and $\bar{D} = \bar{\partial} + \frac{1}{2}B\bar{\partial} \ln g_{z\bar{z}}$ and we take the electron mass $m = 2$ for simplicity. The ground-state wavefunction satisfies

$$\bar{D}\psi = 0 \tag{6}$$

with solution $\psi(z, \bar{z}) = {g_{z\bar{z}}}^{-B/2} F(z) = y^B F(z)$, where $F(z)$ is a holomorphic function. The behaviour of $F(z)$ under Fuchsian transformation were discussed in [8]. But here we want to solve this problem in the context of CFT, so we need to find the behaviour of the wavefunction under a larger transformation, i.e. the general conformal transformation of which the Fuchsian transformation is a subclass. To do so, we note that for a two-dimensional surface with a Poincaré metric, the conformal transformation, which leaves the metric invariant up to a scale change

$$g_{\tilde{z}\bar{\tilde{z}}}\, \mathrm{d}\tilde{z}\, \mathrm{d}\bar{\tilde{z}} = \Omega g_{z\bar{z}}\, \mathrm{d}z\, \mathrm{d}\bar{z} \tag{7}$$

reduces to the analytic coordinate transformations

$$\tilde{z} = f(z) \qquad \bar{\tilde{z}} = \bar{f}(\bar{z}) \tag{8}$$

where f ($\bar{f}$) is a holomorphic (antiholomorphic) function. Under conformal transformation D and $\bar{D}$ change as

$$\tilde{D} = \frac{\mathrm{d}z}{\mathrm{d}\tilde{z}} U^{-1} D U \qquad \tilde{\bar{D}} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}\tilde{\bar{z}}} U'^{-1} \bar{D} U' \tag{9}$$

where

$$U(z,\bar{z}) = \Omega^{-B/2} \left(\frac{\mathrm{d}z}{\mathrm{d}\tilde{z}}\right)^{-B/2} \left(\frac{\mathrm{d}\bar{z}}{\mathrm{d}\tilde{\bar{z}}}\right)^{B/2} \qquad U'(z,\bar{z}) = \Omega^{B/2} \left(\frac{\mathrm{d}z}{\mathrm{d}\tilde{z}}\right)^{-B/2} \left(\frac{\mathrm{d}\bar{z}}{\mathrm{d}\tilde{\bar{z}}}\right)^{B/2}. \tag{10}$$

The Hamiltonian (5) in the new coordinate is

$$H = -g^{\tilde{z}\tilde{\bar{z}}} \tilde{D} \tilde{\bar{D}} + B/4 \tag{11}$$

and the transformed ground-state wavefunction $\tilde{\psi}$ satisfies in $\tilde{\bar{D}}\tilde{\psi} = 0$. Using (6) we find $\psi = U'\tilde{\psi}$, and then using (10) we obtain

$$\tilde{\psi} \Omega^{B/2} \, \mathrm{d}\tilde{z}^{B/2} \, \mathrm{d}\tilde{\bar{z}}^{-B/2} = \psi \, \mathrm{d}z^{B/2} \, \mathrm{d}\bar{z}^{-B/2}. \tag{12}$$

Now considering the decomposition $\psi(z,\bar{z}) = y^B F(z)$, and putting it in (12), we find that $F(z)$ must be a primary field of weight B, i.e. a B-form under a general conformal transformation.

As mentioned in the introduction, the authors of [8] found the holomorphic part of Landau and Laughlin wavefunction by lengthy calculations. Here we want to calculate these functions by using the plasma analogy of the QHE, i.e. again using the Green functions of the Coulomb gas, but now on a Riemann surface. So let us first give a brief review of the Coulomb gas model on a Riemann surface. This model is defined by a bosonic scalar field coupled to a background charge Q and is described by the following action [15]:

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z \left(\partial\Phi(z,\bar{z}) \bar{\partial}\Phi(z,\bar{z}) + \frac{1}{4} Q \sqrt{g} R \Phi(z,\bar{z}) \right) \tag{13}$$

where R is the scalar curvature of the surface and $g = \det g_{\mu\nu}$. In what follows we shall consider only the holomorphic part of the correlation functions, and hence we require φ (the holomorphic part of Φ) to compactify on a unit circle $R/2\pi Z$ [15]. R is the real line and Z denotes integer numbers. The correlation function of the vertex fields $\langle \prod_{j=1}^{N} : \mathrm{e}^{\mathrm{i}\alpha_j \varphi(z_j)} : \rangle$ has been calculated in different contexts [12–15]. In [12] it was obtained by successive application of the Wick theorem and by considering the effect of zero modes. In [13] it was shown that this correlation function can be derived by splitting $\varphi(z)$ to its zero- and non-zero-mode components

$$\varphi(z) = 2\pi \sum_{i=1}^{g} p_i \int \omega_i(\nu) \, \mathrm{d}\nu + \hat{\varphi}(z) \tag{14}$$

where p_i and $\hat{\varphi}(z)$ are independent free fields and p_i are zero-mode oscillators. The contraction rule for $\hat{\varphi}(z)$ is $\langle \hat{\varphi}(z)\hat{\varphi}(w) \rangle = -\ln E(z,w)$ [13, 15], which is the Green function of two charges located at z and w, interacting via a Coulomb potential in two dimensions. In [14] the correlation function was obtained by using the b–c system, which is described by the first-order action $S = \int \mathrm{d}^2 z \, b\bar{\partial}c$. b and c are conformal fields with weights λ and $\lambda - 1$, respectively. By calculating the correlation function of the vertex fields (vertex insertions), the authors of [14] showed that they are the same as the corresponding one in the Coulomb gas model. The result obtained in all the above papers is

$$\left\langle \prod_{j=1}^{N} V_{q_j}(z_j) \right\rangle = \left\langle \prod_{j=1}^{N} : \mathrm{e}^{\mathrm{i}q_j \varphi(z_j)} : \right\rangle = \prod_{k=1}^{N} \sigma^{Q q_k}(z_k) \prod_{i<j}^{N} E^{q_i q_j}(z_i, z_j) \theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (cv|\mathrm{d}\Omega). \tag{15}$$

Here $\sigma(z)$ is a holomorphic $(g/2, 0)$-form, without zero or pole, where g is the genus of the surface and $\sigma(g = 1) = 1$. $E(z_i, z_j)$ is a holomorphic $(-1/2, 0)$-form, which is antisymmetric under the interchange of its coordinates and is zero for $z_i = \gamma(z_j)$; $\gamma \in \Gamma$, $\Gamma \subset$ PSL(2R). $v = \sum_i q_i z_i - Q\Delta$, in which the Riemann class Δ is a $(g-1)$ degree divisor. The theta characteristics (δ, ϵ), c and d must be consistent with the boundary condition of $V_{q_i}(z_i)$. By boundary condition we mean the behaviour of $\langle V_{q_i}(z_i)\rangle$ under the winding of the point z_i around the homology cycles of our Riemann surface. It can also be shown that the correlation function (15) vanishes, unless the total charges q_i cancel the background charge Q [12]:

$$\sum_i q_i = -\frac{Q}{8\pi}\int \mathrm{d}^2 z\,\sqrt{g}R(z) = Q(g-1). \tag{16}$$

Now if we want the correlation function (15) to describe a fermionic wavefunction, $q_i q_j$ must be an odd integer (as $E(z_i, z_j)$ is antisymmetric). Also if we demand that all fermions are identical, we must choose all q_i to be equal to $\sqrt{m}$, where m is an odd integer. In this way $\left\langle\prod_{i=1}^N V_{q_i}(z_i)\right\rangle$ becomes a Jastrow-type wavefunction.

Another necessary condition for $\left\langle\prod_{i=1}^N V_{q_i}(z_i)\right\rangle$ to be a Laughlin wavefunction, is that its behaviour under the action of conformal group transformation must be consistent with the conformal weight of electrons wavefunctions, which, as mentioned after (12), is equal to B. Now as the conformal weight of $\mathrm{e}^{\mathrm{i}q\varphi}$ is $\frac{1}{2}q(q+Q) = \frac{1}{2}\sqrt{m}(\sqrt{m}+Q)$, we obtain

$$B = \tfrac{1}{2}(m + \sqrt{m}Q). \tag{17}$$

Following the above discussion, the appropriate wavefunction for a Laughlin state, which satisfies (12) and (16) for each of its coordinates, is

$$\begin{aligned}\psi(z_1,\ldots,z_N) &= \prod_{i=1}^N y_i^B \left\langle \prod_{j=1}^N V_{q_j}(z_j)\right\rangle \\ &= \prod_{i=1}^N y_i^B \prod_{i=1}^N \sigma^{2B-m}(z_i) \\ &\quad\times \prod_{i<j}^N E^m(z_i,z_j)\theta\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}\left(m\sum_{i=1}^N z_i - (2B-m)\Delta|m\Omega\right).\end{aligned} \tag{18}$$

Following the freedom of choosing the characteristics of the theta function of (15) (as discussed in [14]), we can choose $c = \sqrt{m}$ and $d = m$ in our case. This choice of c and d is consistent with the wavefunction on the torus [5, 11] and also ensures that the phase of ψ, when the points z_i wind around the homology cycles, does not depend on z_i. This independence comes from the invariance of (6) under this winding [8]. Now what are the characteristics δ and ϵ? As discussed in [8], by comparing the behaviour of the wavefunction under Fucshian transformations $z \longrightarrow \gamma z$, with the behaviour of the theta functions under similar transformations, one arrives at $\delta = \delta_0 + l/m$, $(l_i = 1,\ldots,m$ and $i = 1,\ldots,g)$ and $\epsilon = \epsilon_0$, where δ_0 and ϵ_0 are g-component constant vectors with components in the interval $[0, 1)$. These values of δ and ϵ give the correct degeneracy number of the Laughlin wavefunctions, i.e. $mg - g + 1$. The explicit values of δ_0 and ϵ_0 depend on our explicit choice of the phase which appears from wavefunction under $z \longrightarrow \gamma z$. For example, in [5, 24] these values were fixed by choosing $u(\gamma_j, z_j) = \mathrm{e}^{\mathrm{i}\phi_j}$ $(j = 1,\ldots,2g)$, where $u(\gamma, z)$ is defined through $\psi(\gamma z) = u(\gamma, z)\psi(z)$, γ_j is a transformation identifying the sides in the fundamental polygon which represents our Riemann surface in the covering space, and ϕ_j is the flux through the jth cycle. In this way our final result (18) becomes exactly the same as that obtained in [8].

Now let us investigate the plasma description of the the wavefunction (18). We write this wavefunction as $\psi = \psi_1\psi_2$ where

$$\psi_1 = \prod_{i=1}^{N} y_i^B \prod_{i<j}^{N} E^m(z_i, z_j) \tag{19}$$

and

$$\psi_2 = \prod_{i=1}^{N} \sigma^{2B-m}(z_i)\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} \left(m \sum_{i=1}^{N} z_i - (2B-m)\Delta | m\Omega \right). \tag{20}$$

ψ_1 only depends on the interaction part of the wavefunction, i.e. the Coulomb interaction, and ψ_2 is related to the spin structure of the electron wavefunction on this surface. Therefore the interaction potential U, which can be defined as

$$|\psi_1|^2 = \mathrm{e}^{-\beta U} \tag{21}$$

becomes

$$U = -\frac{1}{\beta}\left(\sum_{i=1}^{N} \ln {y_i}^{2B} + \sum_{i<j}^{N} \ln |E(z_i - z_j)|^{2m} \right). \tag{22}$$

Now as in the fundamental domain of the Riemann surface we have

$$\partial_z \partial_{\bar{z}} \ln |E(z, w)|^{2m} = \pi m \delta^2(z - w) \tag{23}$$

by choosing $1/\beta = m$, we see that U is a Coulomb potential of particles with charge m, interacting with themselves and with a uniform background charge $\rho_0 = B/2\pi$ [7]. Charge neutrality of the plasma requires that the plasma particles spread out in the surface with density $\rho_m = \rho_0/m$, which corresponds to a filling factor $\nu = 1/m$.

To determine the precise value of m, we use equations (16) and (17) to obtain

$$m(N + g - 1) = 2B(g - 1). \tag{24}$$

This equation gives the value of m in terms of the magnetic field B, the genus g, and the number of electrons N. It is also interesting to see the geoemetrical meaning of (24). By the Riemann vanishing theorem, the number of zeros of the theta function of (18) is mg and as $\prod_{i<j}^{N} E(z_i, z_j)^m$ (as a function of z_i) has $m(N-1)$ zeros, so the number of zeros of the wavefunction (18) with respect to each of its coordinates is $mg + m(N-1)$, which from (24) is equal to the magnetic flux $\phi = 2B(g-1)$. This shows that the degree of the line (for $m = 1$) or vector (for $m > 1$) bundle is equal to the first Chern number of the gauge field, as expected.

As a last point, we know that by suitable choice of Q the Coulomb gas model can be considered as a set of minimal models. The minimal models are characterized by two positive coprime integers p and q with central charge $c(Q) = 1 - 6(p-q)^2/pq$. Now as the central charge of a Coulomb gas is $c(Q) = 1 - 3Q^2$, equation (17) shows that if B and m satisfy

$$\left(\frac{2B - m}{\sqrt{m}} \right)^2 = \frac{2(p-q)^2}{pq} \tag{25}$$

our QHE is a (p, q) minimal model. For example, for $p = q + 1$ unitary minimal models, any odd integer m which satisfies

$$m = \frac{q(q+1)}{2} \left(\frac{r}{s} \right)^2 \tag{26}$$

where r and s are integers, has a $(q+1, q)$ corresponding minimal model description. At $Q = 0$, c is equal to 1 and B is $m/2$. A detailed discussion of this case can be found in [18].

4. Quasiholes on a Riemann surface

In this section we want to study the aspect of quasihole states in the context of conformal field theory. Laughlin argued that the lowing excited state of the QHE are produced by creation of quasiparticles (quasiholes) in the system. These are particles that obey fractional statistics, i.e. by interchanging two of them, the wavefunction takes the $e^{i\theta}$ phase. This phase for quasiholes is π/m where $\nu = 1/m$ is the filling factor [1, 17]. If we want to express these particles in terms of vertex fields, we must choose the appropriate charges for these vertices. Using (15) it can be seen that by interchanging two vertices we obtain

$$\langle V_{q_i}(z_i)V_{q_j}(z_j)\rangle = e^{i\pi q_i q_j}\langle V_{q_i}(z_j)V_{q_j}(z_i)\rangle \tag{27}$$

so we must choose $q_i = 1/\sqrt{m}$ to relate the vertex fields to the quasiholes. Now consider a system containing N electrons (represented by vertices with charge $\sqrt{m}$), and N_q quasiholes (represented by vertices with charges $1/\sqrt{m}$), then equation (16) leads to

$$N\sqrt{m} + \frac{N_q}{\sqrt{m}} = Q(g-1). \tag{28}$$

Using equations (16) and (17) (which also holds in this case), we determine the filling factor

$$m(N+g-1) + N_q = 2B(g-1) = \phi. \tag{29}$$

This relation is consistent with the result pointed out in [17], and can be used to obtain the electric charge of quasiholes with the method that was introduced in [4]. If the system of N electrons, in the Laughlin state, is excited at fixed magnetic field by removal of an electron, the final state has the following flux:

$$\phi(N; m) = \phi(N-1; m) + m \tag{30}$$

where $\phi(N; m) = m(N+g-1)$. Comparison of equations (29) and (30) shows that the new system (30) is composed of $N-1$ electrons and m quasiholes. Hence the quasiholes carry the charge $e^* = e/m$ ($e > 0$). To reproduce this result in another way, we note that the charge of particles can also be determined by using the OPE of the current and corresponding fields [9]. On a Riemann surface, the above OPE [19] is

$$J(z)e^{i\varphi(w)/\sqrt{m}} = \frac{1/m}{z-w}e^{i\varphi(w)/\sqrt{m}} + \cdots. \tag{31}$$

Therefore following the steps of [9], the charge of the quasihole corresponding to the vertex $e^{i\varphi(w)/\sqrt{m}}$ is $e^* = e/m$.

At the end, we present an expression for the holomorphic part of the wavefunction containing N electrons and one quasihole:

$$\begin{aligned}\psi(z, z_1, \ldots, z_N) &= \left\langle V_q(z)\prod_{i=1}^{N} V_{q_i}(z_i)\right\rangle \\ &= \sigma^{(2B-m)/m}(z)\prod_{i=1}^{N}\sigma^{2B-m}(z_i) \\ &\quad\times\prod_{i=1}^{N}E(z_i, z)\prod_{i<j}^{N}E^m(z_i, z_j)\theta\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}\left(m\prod_{i=1}^{N}z_i + z - (2B-m)\Delta|m\Omega\right)\end{aligned} \tag{32}$$

which is obtained by (15). Factorizing this wavefunction as $\psi = \psi_1\psi_2$, where

$$\psi_1 = \prod_{i=1}^{N}E(z_i, z_j)\prod_{i<j}^{N}E^m(z_i, z_j) \tag{33}$$

and ψ_2 other terms, and again by considering $|\psi_1|^2 = \exp(-U/m)$, one can see that U is the Coulomb potential of a system of particles of charge m, interacting with themselves and with a particle of charge 1 located at z (which again proves that $e^* = e/m$).

At the end we would like to add a point. One of the important points in the physics of the QHE is to understand the incompressibility feature of the Laughlin wavefunctions, which may be related to the quantum group symmetry of the Laughlin states [7,20–22]. On the other hand, there is a deep connection between the conformal and quantum group symmetries [23]. Our procedure in expressing the Laughlin states in the context of CFT may shed some light on these connections on Riemann surfaces. We will discuss these elsewhere.

5. Conclusion

Using the analogy of the Coulomb gas and the plasma description of the quantum Hall effect (QHE), the conformal symmetry of Laughlin states, and the results found for the Coulomb gas model on a Riemann surface, we obtained the Laughlin wavefunction on an arbitrary compact and orientable Riemann surface. We also determined the filling factor and degeneracy of these wavefunctions. In the case of the Poincaré metric, we found the plasma description of the QHE on these surfaces and also state the relation between the FQHE and minimal models. Finally, for the cases where the quasiholes are also present, we found the wavefunctions.

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